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Title: NATURAL BASIS FOR SOLUTIONS TO POLYNOMIAL COEFFICIENT ORDINARY DIFFERENTIAL EQUATIONS

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Natural Basis for Solutions to Polynomial Coefficient Ordinary Differential Equations

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1 Abstract

We describe an algorithm which gives a natural basis for the power series solutions for a large class of polynomial coefficient ordinary differential equation of homogeneous order.

2 Introduction

We describe an algorithm which gives a natural basis for the power series solutions for a large class of polynomial coefficient ordinary differential equation of homogeneous order. By a natural basis we mean that any element in the nullspace can be written in the simple form $f(x) = \Sigma_{0\leq j\leq h-1} \frac{[D^{(j)}f(x)]|_0}{j!} e_j(x)$ (where $D^{(j)} = \frac{d^j}{dx^j}$) and where $e_j(x), 0 \leq j \leq h-1$ is the basis. This form will be valid for a large class of polynomial coefficient ordinary differential equations that we consider.

In contrast, the standard method for describing a basis for even the simplist linear homogeneous order ordinary differential equation $\Sigma_{0\leq k\leq h} a_k y^{(k)} = 0$ (where a_k are constants) is written in a form (using roots of the characteristic polynomial) which cannot readily be used to describe an arbitrary element of the nullspace.

In fact we solve a more general problem which deals with noncommutative linear operators (of shift type) defined on a linear space with countably infinite basis. A special case of this more general problem and its solution will be that described above.

A variety of books and papers discuss solutions of other special cases of linear polynomial coefficient ordinary differential equations. To a large extend these special cases have solutions which can be expressed in closed form (as is the standard method for describing a basis of solutions to linear constant coefficient ordinary differential equations). For a survey of general methods see, for example, [1], Ch.1-5;[2],Ch.3; [5],Ch.3; and [3].

3 A Nullspace Problem on an Infinite Dimensional Linear Space

Let U be a linear space with basis $u_k, k \in \mathbf{N} \equiv \{0,1,2,...\}$. So $U = Span\{u_k\} \equiv \{\Sigma_{k \in \mathbf{N}} a_k u_k | a_k \in \mathbf{R}\}$ (where $\mathbf{R}$ is the field over which U is defined) . By a *basis* we mean that $\Sigma_{k \in \mathbf{N}} a_k u_k = 0$ implies that $a_k = 0$ for all k. We also define the projection operators $\pi_j : U \to \mathbf{R}$ by $\pi_j(\Sigma_{k \in \mathbf{N}} a_k u_k) = a_j$ (for $j \in \mathbf{N}$). The π_j are well defined since the u_k form a basis. So we have

$$g = \Sigma_{j=0}^{\infty} \pi_j(g) u_j$$

for $g \in U$.

We consider the shift linear operators $\mu^j : U \to U$ $(0 \le j \le h)$ defined by $\mu^j(\Sigma_{k \in \mathbf{N}} a_k u_k) = \Sigma_{k \in \mathbf{N}} a_k \sigma_{j,k} u_{k-j}$ (where $u_i \equiv 0$ for $i \le 0$ and where $\sigma_{j,k} \in \mathbf{R}$). (Note that these operators do not commute unless $\sigma_{j,k}\sigma_{i,k-j} = \sigma_{i,k}\sigma_{j,k-i}$ for all i, j, k.)

The problem we set ourselves is: Given the linear operator $P(\mu) = \mu^h - \Sigma_{0 \le j \le h-1} c_j \mu^j, c_j \in \mathbf{R}$ which operates on U find a basis for $ker(P(\mu)) \equiv \{f \in U | P(\mu) f = 0\}$.

To solve this problem we will require that

$$\sigma_{k,k+h} \neq 0$$

for $k \ge 0$. We also need to use two auxilliary linear spaces. One of these spaces, V, will have basis $v_k, k \in \mathbf{N}$ and linear operators $\nu^j : V \to V$ given by

$$\nu^j(\Sigma_{k \in \mathbf{N}} b_k v_k) = \Sigma_{k \in \mathbf{N}} b_k \sigma_{j,j+k} v_{j+k}. \tag{1}$$

for $0 \le j \le h$. The other auxiliary linear space W having basis $\alpha_0, ..., \alpha_{h-1}$ is defined as follows:

Let $\phi : V \to W \equiv Span\{\alpha_0, ..., \alpha_{h-1}\}$ where

$$\phi(v_j) \equiv \alpha_j, 0 \le j \le h-1, \tag{2}$$

$$\phi(v_h) \equiv \Sigma_{0 \le j \le h-1} c_j \frac{\sigma_{j,j}}{\sigma_{h,h}} \phi(v_j) = \Sigma_{0 \le j \le h-1} c_j \frac{\sigma_{j,j}}{\sigma_{h,h}} \alpha_j, \tag{3}$$

and for $k \ge 1$ we recursively define

$$\phi(v_{h+k}) \equiv \Sigma_{0 \le j \le h-1} c_j \frac{\sigma_{j,j+k}}{\sigma_{h,h+k}} \phi(v_{j+k}). \tag{4}$$

Hence for $k \in \mathbf{N}, 0 \le j \le h-1$ there exist constants $d_{k,j}$ so that

$$\phi(v_k) = \Sigma_{0 \le j \le h-1} d_{k,j} \alpha_j. \tag{5}$$

We extend the meaning of π_k to W (for $0 \leq k \leq h-1$) by setting $\pi_k(\Sigma_{0\leq j\leq h-1} b_j \alpha_j) \equiv b_k$.

Then by (2) we have

$$d_{k,j} = \pi_k(\phi(v_j)) = \delta_{k,j} \tag{6}$$

for $0 \leq j, k \leq h-1$ where $\delta_{k,j}$ is the Kronecker δ.

Defining $P(\nu) \equiv \nu^h - \Sigma_{0\leq j\leq h-1} c_j \nu^j$ we note that by our construction we have

$$\phi(P(\nu)v_k) = \phi(\nu^h - \Sigma_{0\leq j\leq h-1} c_j \nu^j)v_k = 0. \tag{7}$$

for $k \geq 0$.

Define

$$e_j \equiv \Sigma_{k=0}^{\infty} d_{k,j} u_k$$

(for $0 \leq j \leq h-1$) and note that by (2) and (6) we have

$$\pi_k(e_j) = \delta_{k,j}.$$

Our main theorem is:

THEOREM:

$i)$ The e_j are linearly independent and $P(\mu)e_j = 0$ for $0 \leq j \leq h-1$,

$ii)$ for $f \in ker(P(\mu))$ we have $f = \Sigma_{j=0}^{h-1}(\pi_j f)e_j$,

and

$iii)$ $ker(P(\mu))$ has dimension h.

Before proving our theorem we note the following calculation which relates the operators $P(\mu)$ and $P(\nu)$ via a tensor product on the linear spaces U and V.

For $0 \leq j \leq h$ we have

$(\mu^j \otimes 1)\Sigma_{k=0}^{\infty} u_k \otimes v_k = \Sigma_{k=0}^{\infty} \sigma_{j,k} u_{k-j} \otimes v_k = \Sigma_{k=j}^{\infty} \sigma_{j,k} u_{k-j} \otimes v_k = \Sigma_{k=0}^{\infty} \sigma_{j,k+j} u_k \otimes v_{k+j} = \Sigma_{k=0}^{\infty} u_k \otimes \sigma_{j,k+j} v_{k+j} = (1 \otimes \nu^j)\Sigma_{k=0}^{\infty} u_k \otimes v_k$.

More generally we have

$(Q(\mu) \otimes 1)\Sigma_{k=0}^{\infty} u_k \otimes v_k = (1 \otimes Q(\nu))\Sigma_{k=0}^{\infty} u_k \otimes v_k$ where $Q(\mu) = \Sigma_{j=0}^{h-1} b_j \mu^j$ and $Q(\nu) = \Sigma_{j=0}^{h-1} b_j \nu^j$ for $b_j \in \mathbf{R}$.

PROOF of THEOREM:

First we show that $P(\mu)e_j = 0$ for $0 \leq j \leq h-1$, then we show that every solution can be written in the form $\Sigma_{j=0}^{h-1}(\pi_j f)e_j = 0$.

Now $(P(\mu)\otimes\phi)\Sigma_{k=0}^{\infty} u_k \otimes v_k = (1\otimes\phi)\Sigma_{k=0}^{\infty} u_k \otimes P(\nu)v_k = \Sigma_{k=0}^{\infty} u_k \otimes \phi P(\nu)v_k = 0$ by (7).

Hence

$0 = (P(\mu)\otimes 1)\Sigma_{k=0}^{\infty} u_k \otimes \phi v_k = (P(\mu)\otimes 1)\Sigma_{k=0}^{\infty} u_k \otimes (\Sigma_{j=0}^{h-1} d_{k,j}\alpha_j) = (P(\mu)\otimes 1)\Sigma_{j=0}^{h-1} e_j \otimes \alpha_j = 0$ which implies that $P(\mu)e_j = 0$ for $0 \leq j \leq h-1$

since (2) implies that $\alpha_0, ..., \alpha_{h-1}$ are linearly independent.

Now we show that any solution f can be written in the form $\Sigma_{j=0}^{h-1}(\pi_j f)e_j$.

Before proceeding we need two new notations. For $g = \Sigma_{k=0}^{\infty} b_k u_k \in U, (g \neq 0)$ we write

$$lt(g) = b_m$$

if $b_m \neq 0$ and if $b_k = 0$ for $k < m$. If $lt(g) = b_m$ we write

$$slt(g) = m.$$

lt stands for "least term" and slt stands for "subscript of the least term". Let $f \in ker(P(\mu))$ and set

$$\hat{f} = f - \Sigma_{j=0}^{h-1}(\pi_j f)e_j.$$

We show that $\pi_k(\hat{f}) = 0$ for all k.

For $k \in \{0, 1, ..., h-1\}$ we have $\pi_k \hat{f} = \pi_k f - \Sigma_{j=0}^{h-1}(\pi_j f)\pi_k e_j = \pi_k f - \Sigma_{j=0}^{h-1}(\pi_j f)\delta_{k,j} = 0$. So $\hat{f} = \Sigma_{k=h}^{\infty} a_k u_k$ where $a_k \in \mathbf{R}$. Now either $\hat{f} = 0$ or $lt(\hat{f}) = a_m$ some $m \geq h$.

So we suppose, by way of contradiction that $lt(\hat{f}) = a_m$, $m \geq h, a_m \neq 0$ so that $\hat{f} = \Sigma_{k=m}^{\infty} a_k u_k$.

CLAIM: $lt(P(\mu)\hat{f}) \neq 0$. (From which we get the contradiction $\hat{f} \notin ker(P(\mu))$).

Now

$P(\mu)\hat{f} = (\mu^h - \Sigma_{0\leq j\leq h-1} c_j \mu^j)(\Sigma_{k=m}^{\infty} a_k u_k) = a_m \mu^h u_m + \mu^h(\Sigma_{k=m+1}^{\infty} a_k u_k) - (\Sigma_{0\leq j\leq h-1} c_j \mu^j) a_m u_m - (\Sigma_{0\leq j\leq h-1} c_j \mu^j)(\Sigma_{k=m+1}^{\infty} a_k u_k) =$

$a_m \sigma_{h,m} u_{m-h} + \Sigma_{k=m+1}^{\infty} a_k \sigma_{h,k} u_{k-h} - \Sigma_{j=0}^{h-1} c_j a_m \sigma_{j,m} u_{m-j} - \Sigma_{j=0}^{h-1} \Sigma_{k=m+1}^{\infty} c_j a_k \sigma_{j,k} u_{k-j}$.

So $slt(\Sigma_{k=m+1}^{\infty} a_k \sigma_{h,k} u_{k-h}) \geq m + 1 - h > m - h$,

$slt(\Sigma_{j=0}^{h-1} c_j a_m \sigma_{j,m} u_{m-j}) \geq m - (h-1) > m - h$, and

$slt(\Sigma_{j=0}^{h-1} \Sigma_{k=m+1}^{\infty} c_j a_k \sigma_{j,k} u_{k-j}) \geq (m+1) - (h-1) > m - h$.

Hence $lt(P(\mu)(\Sigma_{k=m}^{\infty} a_k u_k)) = a_m \sigma_{h,m} \neq 0$.

Lastly, since the e_j are linearly independent and $ker(P(\mu)) = span\{e_j | 0 \leq j \leq h-1\}$ we have that $(ker(P(\mu)))$ has dimension h.

END of PROOF.

The iteration process described by (3) and (4) can be used to find $d_{j,k}$ by using a matrix product of h by h matrices.

Let

$$M(k+h) \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & c_0 \frac{\sigma_{0,k}}{\sigma_{h,k+h}} \\ 1 & 0 & 0 & \cdots & 0 & 0 & c_1 \frac{\sigma_{1,k+1}}{\sigma_{h,k+h}} \\ 0 & 1 & 0 & \cdots & 0 & 0 & c_2 \frac{\sigma_{2,k+2}}{\sigma_{h,k+h}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & c_{h-1} \frac{\sigma_{h-1,k+h-1}}{\sigma_{h,k+h}} \end{bmatrix}$$

Then

$$(\phi(v_{k+1}), \phi(v_{k+2}), \cdots, \phi(v_{k+h})) = (\alpha_0, \alpha_1, \cdots, \alpha_{h-1}) M(h) \cdots M(k-1+h) M(k+h) =$$

$$(\alpha_0, \alpha_1, \cdots, \alpha_{h-1}) \begin{bmatrix} d_{k+1,0} & \cdots & d_{k+h,0} \\ \vdots & \vdots & \vdots \\ d_{k+1,h-1} & \cdots & d_{k+h,h-1} \end{bmatrix}$$

4 Application to Polynomial Coefficient ODE's

Let $R[x, D]$ denote the ring of polynomials in x and $D \equiv \frac{d}{dx}$. The elements of $R[x, D]$ act as operators on $R[[x]]$ the ring of formal power series in x.

$R[x, D]$ is noncommutative since

$$Dx = 1 + xD \tag{8}$$

which follows from the fact that for $f \in R[[x]]$ we have $(Dx)f = D(xf) = f+xD(f) = (1+xD)f$. This algebra and its generalization to partial differential equations are called *Weyl Algebras.*

We will call $x^k D^m$,$(k, m \in \mathbf{N})$ a *reduced term* of $R[x, D]$. By (8), any term in $R[x, D]$ can be written as a linear combination of reduced terms. In particular we will assume that any polynomial $P(x, D) \in R[x, D]$ is written as a linear combination of reduced terms.

For $j \in \mathbf{Z}$ define $F_j = Span_R\{x^q D^m | q - m = j\}$. Then $R[x, D] = \oplus_{-\infty}^{\infty} F_j$.

So we will write any polynomial in $R[x, D]$ in the form

$$P(x, D) = P_h(x, D) - \Sigma_{0 \leq j \leq h-1} P_j(x, D)$$

where $P_j(x, D) \in F_j$.

We are now interested in how an arbitrary element

$$Q_j = \Sigma_{[q-m=j]} b_{j,q,m} x^q D^m \in F_j$$

(for $j \in \mathbf{Z}$) operates on $R[[x]]$.

Now

$$x^q(x^k) = x^{q+k},$$

$$D^m(x^k) = \frac{k!}{(k-m)!} x^{k-m} \text{ if } (k-m) \in \mathbf{N},$$

and

$$D^m(x^k) = 0 \text{ if } (k-m) \notin \mathbf{N}.$$

Using the notation $c(m, k) \equiv \frac{k!}{(k-m)!}$ if $(k - m) \geq 0$ and $c(m, k) \equiv 0$ if $(k - m) < 0$ we get that

$$x^q D^m(x^k) = c(m, k) x^{k-m+q}.$$

So

$$Q_j(x^k) = \Sigma_{[m=k-j]} b_{j,k,m} c(m, k) x^{k-j}.$$

Hence there exist constants $\sigma_{j,k} \equiv \Sigma_{[m=k-j]} b_{j,k,m} c(m,k) \in \mathbf{R}$ so that

$$Q_j(x^k) = \sigma_{j,k} x^{k-j}.$$

REMARK: Various properties of the general Weyl Algebra are discussed in Chapter 1 of the excellent book [4].

5 An Algorithm for finding a Basis for $ker(P(x,D))$

Let $P(x,D) = P_h(x,D) - \Sigma_{0 \leq j \leq h-1} P_j(x,D)$ where $\sigma_{j,k}$ are defined so that $P_j(x,D)x^k = \sigma_{j,k} x^{k-j}$ for $P_j(x,D) \in F_j$. We also require that and $\sigma_{h,h+k} \neq 0$ for $k \geq 0$.

Let $U = Span\{u_k\}$, $P(\mu) = \mu^h - \Sigma_{0 \leq j \leq h-1} \mu^j$ where $\mu^j(\Sigma_{k=0}^{\infty} a_k u_k) = \Sigma_{k=j}^{\infty} \sigma_{j,k} a_k u_{k-j}$ and where $\sigma_{j,k}$ are defined above.

The algorithm in Section 3 shows how to construct a set of h solutions $e_j = \Sigma_{0 \leq k \leq \infty} d_{k,j} u_k, 0 \leq j \leq h-1$ so that

$$i)\ e_j \text{ are linearly independent and } P(\mu)e_j = 0,$$

$$ii)\ \text{any } f \in ker(P(\mu)) \text{ can be written in the form } f = \Sigma_{0 \leq j \leq h-1} (\pi_j f) e_j.$$

Another conclusion from Section 3 is that

$$ker(P(\mu)) \text{ has dimension } h.$$

The algorithm also shows how to compute the $d_{k,j}$ via a matrix product.

Consider the R linear isomorphism $\Psi : U \to R[[x]]$ given by

$$\Psi(u_k) = x^k,$$

and

$$\Psi(\mu^j) = P_j(x,D)$$

for $k \in \mathbf{N}$ and $0 \leq j \leq h$.

So

$$\Psi(P(\mu)) = \Psi(\mu^h - \Sigma_{0 \leq j \leq h-1} \mu^j) = P_h(x,D) - \Sigma_{0 \leq j \leq h-1} P_j(x,D).$$

Then for $g \in U$ we have

$$\Psi(P(\mu)) = \Psi(P(\mu))\Psi(g)$$

and we have that

$$\Psi : ker(P(\mu)) \to ker(P(x,D))$$

is one to one and onto.

Set

$$e_j(x) \equiv \Psi(e_j) = \Sigma_{k=0}^{\infty} d_{k,j} x^k$$

for $0 \leq j \leq h-1$.

Since $\frac{D^{(j)}}{j!}(\Sigma_{k=0}^{\infty}a_k x^k)|_0 = a_j$ we see that

$$\Psi(\pi_j f) = [\frac{D^{(j)}}{j!}(\Psi(f))]|_0$$

for any $f \in U$.

Hence

$ker(P(x,D)) = \{f(x) \in R[[x]]|P(x,D)f(x) = 0\}$ has dimension h and the algorithm in the Section 3 shows how to compute a basis e_j (and consequently $e_j(x)$) having the property that if $f(x) \in ker(P(x,D))$ then

$$f(x) = \Sigma_{0 \leq j \leq h-1}\frac{D^{(j)}(f(x))|_0}{j!}e_j(x).$$

REMARK: The construction in Section 3 should extend to give a basis for the case where $\mu^h - \Sigma_{l \leq j \leq h-1}c_j\mu^j$ and $l < 0, h > 0$. The case (where $\sigma_{h,h+k} = 0$ for some $k \geq 0$) would require a more delicate analysis than the one in this paper since the iteration process would have to be significantly modified.

REMARK: In addition to giving a survey of methods used in analysing and solving ordinary differential equations, the paper [3] also gives an extensive bibliography.

References

[1] E.G.C. Poole, **Introdution to the Theory of Linear Differential Equations**,Oxford University Press, 1936.

[2] L.S.Pontryagin,**Ordinary Differential Equations**,Addison-Wesley Publishing Co.,1962.

[3] M.F. Singer **Formal Solutions of Differential Equations** *J. Symbolic Comput.* **10**, 1990, no. 1, 59-94.

[4] J.E. Björk, **Rings of Differential Operators**,1979, North-Holland Mathimatical Library,21.

[5] E.A.Coddington,N.Levinson,**Theory of Ordinary Differential Equations**,McGraw-Hill Book Company,Inc.1955.